

EXISTENCE OF TRAVELLING WAVES AND HIGH ACTIVATION ENERGY LIMITS FOR A ONEDIMENSIONAL THERMO-DIFFUSIVE LEAN SPRAY FLAME MODEL

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ABSTRACT. We provide a mathematical analysis of a thermo-diffusive combustion model of lean spray flames, for which we prove the existence of travelling waves. In the high activation energy singular limit we show the existence of two distinct combustion regimes with a sharp transition – the diffusion limited regime and the vaporisation controlled regime. The latter is specific to spray flames with slow enough vaporisation. We give a complete characterisation of these regimes, including explicit velocities, profiles, and upper estimate of the size of the internal combustion layer.

Our model is on the one hand simple enough to allow for explicit asymptotic limits and on the other hand rich enough to capture some particular aspects of spray combustion. Finally, we briefly discuss the cases where the vaporisation is infinitely fast, or where the spray is polydisperse.

Keywords: reaction-diffusion equations, spray flames, travelling waves, high activation energy, singular limits

1. INTRODUCTION

This paper provides a rigorous mathematical analysis of some aspects of spray combustion, including the analysis of the so-called high activation energy limit for a spray flame model. This notion of high activation energy limit was first introduced in the pioneering work of Zeldovich and Frank-Kamenetskii [42], and refers to the limit where the combustion rate is much faster than any other physical phenomenon, in particular diffusion. Since then, there have been many studies of these asymptotics and applications to gaseous flames. However, gas-vapor-droplets systems have many applications in industry or everyday's life, such as diesel or propulsion engines. When trying to understand some of their specific features, it appears that the structure of these two-phase flames as well as their speed or stability are greatly affected by the presence of vaporising liquid droplets possibly interacting with the combustion zone.

The behaviour of spray flames has been investigated a lot in the literature and a wide variety of regimes were considered. Dating from the 70s and early 80s, we can quote the works of POLYMEROPOULOS et al. [31][32], MIZUTANI et al. [25][26], HAYASHI et al. [17][18]. BALLAL et al. [2]. They present studies of the propagation

of liquid fuel sprays, study the influence of the size of the droplets, the type and geometry of the spray flame as well as its structure. The investigation of those vapor–drop–air systems was continued in the 90s. To quote but a few works, we refer to the work of AGGARWAL and SIRIGNANO [1] as well as the papers of GREENBERG, TAMBOUR and SILVERMAN [34][35], where the structure of spray flames as well as the influence of parameters such as droplet size, fuel volatility, or equivalence ratio are also investigated analytically. More elaborate situations appearing in propulsion engines are for example pulsating or acoustic instabilities. We refer to [16][15][11][12][13][7] and more recently [8][14][20][21][22] for studies in that direction.

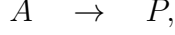
Only few of these studies of spray flames involve rigorous–in–the–mathematical–sense analysis of spray flames models where existence, uniqueness, or asymptotic limits are derived (see e.g. [23, 8]). This is in contrast with purely gaseous combustion, where a lot of results exist in the mathematical literature for various regimes and asymptotics. In particular, a complete study of thermo-diffusive lean gaseous flame fronts and the high activation energy limit is in the paper of BERESTYCKI, NICOLAENKO and SCHEURER [4], and in the paper of BERESTYCKI and LARROUTOU [3]. The originality of the present work is to provide a complete mathematical analysis of the counterpart of those systems, namely a lean **spray** flame model that on the hand is **simple enough** to allow for explicit asymptotic limits, and on the other hand **rich enough** to capture some particular aspects of spray combustion. Those results should be compared with the very interesting work of SUARD, NICOLI and HALDENWANG [38]. Using numerical simulations, these authors investigate the scaling laws of the spray flame with respect to the vaporisation rate of the liquid phase. When the vaporisation is fast enough, the velocity of the spray flame is comparable to that of the gaseous flame where the whole reactant would be present only in a gaseous form. This is the **diffusion controlled** regime. On the contrary when the time for complete vaporisation exceeds a critical value, the velocity of the spray flame starts to decrease. This is the so-called **vaporisation controlled** regime. In this case, they also investigated the structure of the reaction zone, and showed that it has a more complex structure than a gaseous flame.

In this work, we prove the existence of spray flame travelling waves and the existence of **two distinct combustion regimes** with a sharp transition in the **high activation energy (HAE)**. In this limit, we provide a complete characterisation of the profiles, that can be written explicitly provided simple analytic expressions of the vaporisation law. We also show that the internal combustion layer is likely to be much larger in the vaporisation controlled regime as compared to the size of the internal combustion layer of a comparable gaseous flame.

1.1. The equations of the lean spray flame model.

The classical gaseous reactive flow. A gas mixture of total mass density ρ is considered to be made of N fluids corresponding to the different species present in the mixture.

If the mass density of those species are $(\rho_i)_{i=1,N}$ then by definition $\rho = \sum_i \rho_i$, and the respective mass fractions $(Y_i)_{i=1,N}$ of the species are defined by $\rho_i = \rho Y_i$. In the case of a single reactant Y , the chemical reaction writes as



where A denotes the reactant species and P denotes one (or a linear combination) of the species produced by the reaction. Notice that the rate of variation of the single reactant A is directly proportional to minus that of the products and it suffices to determine the mass fraction Y of the reactant A in order to determine the mass fractions of the other species.

The mass, momentum, energy equation for a reactive flow in one dimension without external forces write, setting $D_t := \partial_t + v\partial_x$,

$$\partial_t \rho + \partial_x(\rho v) = D_t \rho + \rho \partial_x v = 0,$$

$$\rho D_t v + \partial_x p = \frac{4}{3} \partial_x (\kappa \partial_x v),$$

$$\rho c_p D_t T - \partial_x (\lambda \partial_x T) = Q \omega + D_t p + \frac{4}{3} \kappa (\partial_x v)^2,$$

where v is the mass-average velocity of the mixture, p the hydrostatic pressure, κ the dynamic viscosity coefficient, T the temperature, c_p the specific heat at constant pressure, $\lambda := \lambda(T)$ the thermal conductivity, Q the chemical heat release of the reaction and ω is the rate at which the reaction occurs and has the form

$$\omega = B(T) \frac{\rho Y}{\mu} \exp \left(-\frac{E}{RT} \right),$$

where E is the activation energy of the single reaction, R the perfect gas constant, $T_A = E/R$ the activation energy temperature, μ the molecular mass of the reactant A , and $B(T)$ a non-stiff prefactor. Finally the equation for the mass fraction Y of the reactant A obeys

$$\rho D_t Y - \partial_x (\gamma \partial_x Y) = -\mu \omega.$$

We refer the reader to [37, 41] for more details.

The liquid phase. In the situation where the reactant A is also present in the mixture in the form of vaporising liquid droplets that are well dispersed in the mixture, one can treat the liquid phase as a new continuous “species”, where the droplets are homogeneously spread inside the mixture, leading to both homogenized vaporisation and combustion in the bulk of the gaseous phase. This is certainly not satisfied in practice, where spray combustion can typically involve flames surrounding individual droplets, with strong temperature gradients within the flame. However, we need this hypothesis for the derivation of our model. Also, ideally the continuity equation for the gaseous mixture should take into account the volume fraction occupied by the liquid phase, the momentum equation should incorporate the drag forces, the energy

equation should include the loss of heat due to the latent heat of the liquid as well as the dissipation due to the drag forces, etc.

We are going to consider regimes in which we can discard those couplings. Since the mass density of the liquid reactant A is in general several orders of magnitude larger than that of the gaseous species, one can assume that the volume fraction of the liquid phase is small. The drag forces are always present and can produce complex turbulent flows. However, the drag force is typically inversely proportional to the surface area of the droplets and proportional to the difference of velocities of the droplets and the surrounding gas. Small enough droplets behave as “passive scalars”, that is particles whose velocity can be set equal to that of the surrounding fluid. The drag forces are internal forces for the two-phase flow and do not affect the total momentum ($\rho_l v_l + \rho v$) of the liquid–gas mixture. As a consequence, since $v_l \simeq v$, one can neglect the drag forces in the gaseous mixture if the mass density ρ_l of the liquid phase is small enough compared to the mass density ρ of the gas mixture. The same conclusion holds for the energy equation.

Let now M denote the mass of an isolated vaporising droplet immersed in a gas. The vaporisation law of the droplet can be very complex (see e.g. [39, 40, 30, 41, 36]). As an approximation, we will assume that the vaporisation rate ϕ only depends upon the temperature T of the surrounding gas and the mass M of the droplet, i.e.

$$d_t M = -\phi(T, M).$$

We will be mainly interested in the monodisperse case, where all droplets in the unburnt gas have the same size. In the laminar flows with constant velocity that we will be considering, this implies that particles located at the same position x at time t all have the same size. We can therefore introduce the mass profile $M(t, x)$ which obeys,

$$D_t M = -\phi(T, M).$$

The situation for polydisperse sprays is briefly considered in the HAE limit in Section 4. Let further $n(t, x)$ be the number density of droplets, that is the number of droplets per unit volume. The mass density of the liquid phase writes $\rho_l(t, x) = n(t, x)M(t, x)$. In the laminar flows we are considering, the droplets do not coalesce nor break-up, hence

$$D_t n = 0,$$

that is the number density of the droplets is simply advected by the flow, and $D_t \rho_l = n(t, x)D_t M$. It follows that the continuity equation for the gaseous reactant A now incorporates the vaporisation flux as a source term

$$\rho D_t Y - \partial_x(\gamma \partial_x Y) = -\mu\omega - D_t \rho_l = -\mu\omega - n D_t M.$$

Finally, we assume that the effects of the latent heat are negligible. This is true when the mass density of the liquid phase is small enough compared to that of the gaseous mixture, which we assumed also in order to neglect the drag forces. The

study of the influence of the latent heat is certainly an important and interesting feature since it can greatly influence the speed or existence of the flame. However this study is beyond the scope of the present work.

Two-phase system. We are interested in laminar, low Mach number, regimes where one can neglect the pressure terms as well as the kinematic viscosity. We want to study the existence and properties of travelling waves moving at constant speed v_0 to the left. In the frame of reference of an observer moving at speed v_0 to the left, the profiles are steady, solution to the system

$$\begin{aligned} v_0 \rho' + (\rho v)' &= 0, \\ \rho v_0 v' &= 0, \\ c_p(\rho v_0 + \rho v)T' - (\lambda T')' &= Q\omega, \\ n' &= 0, \\ (\rho v_0 + \rho v)Y' - (\gamma Y')' &= -\mu\omega - n(v_0 + v)M'. \end{aligned}$$

From the continuity equation, the mass flux $c := \rho(v_0 + v)$ is a constant. Moreover $v' = 0$ implies $v = 0$ in the moving frame, and therefore also $\rho' = 0$, so that the gas density is constant, $\rho := \rho_0$. Setting also $n := n_0$, a constant, the system reduces to

$$\begin{aligned} cc_p T' - (\lambda T')' &= Q\omega, \\ cY' - (\gamma Y')' &= -\mu\omega - c \frac{n_0}{\rho_0} M', \\ c \frac{n_0}{\rho_0} M' &= -n_0 \phi(T, M), \end{aligned}$$

with prescribed values (T_u, Y_u, n_u, M_u) in the unburnt region,

$$T(-\infty) = T_u, \quad Y(-\infty) = Y_u, \quad n_0 = n_u, \quad M(-\infty) = M_u,$$

and assuming complete reaction and complete vaporisation on the burnt region

$$Y(+\infty) = 0, \quad M(+\infty) = 0.$$

The temperature $T(+\infty) := T_b$ in the burnt region is obtained by integrating the equation for the quantity

$$\frac{c_p}{Q}T + \frac{1}{\mu} \left(Y + \frac{n_0}{\rho_0} M \right)$$

on $(-\infty, +\infty)$. It follows

$$T_b = \frac{Q}{c_p \mu} \left(Y_u + \frac{n_0}{\rho_0} M_u \right).$$

Normalized system. The normalized variables are (u, v, m) defined as

$$(1) \quad u = \frac{T - T_u}{T_b - T}, \quad v = \frac{\rho_0 Y}{\rho_0 Y_u + n_0 M_u}, \quad m = \frac{M}{\rho_0 Y_u + n_0 M_u},$$

solution to the system,

$$(2) \quad \begin{aligned} -u'' + cu' &= \tilde{f}(u)v & \text{on } \mathbb{R}, \\ -\Lambda v'' + cv' &= -\tilde{f}(u)v - cn_0 m' & \text{on } \mathbb{R}, \\ cm' &= -\tilde{\phi}(u, m) & \text{on } \mathbb{R}, \\ u(-\infty) &= 0, & u(+\infty) = 1, \\ v(-\infty) &= v_u, & v(+\infty) = 0, \\ m(-\infty) &= m_u, \end{aligned}$$

where $\Lambda = \gamma c_p / \lambda$ is the reciprocal of the Lewis number, and with appropriate renormalized formulas for the reaction rate \tilde{f} and the vaporisation rate $\tilde{\phi}$ (we omit the tildes below). Also $n_0 \geq 0$, $v_u \geq 0$, $m_u \geq 0$ and $v_u + n_0 m_u = 1$.

Reaction and vaporisation laws. In order to avoid the cold boundary effect, we assume that the reaction rate is zero below the normalized ignition temperature $0 < \theta_i < 1$, namely $f : [0, 1] \rightarrow \mathbb{R}^+$, $f(u) = 0$, for all $0 \leq u < \theta_i$, f positive on $(\theta_i, 1]$ and Lipschitz continuous on $[\theta_i, 1]$.

For the study of the high activation energy limit, we will use on $(\theta_i, 1]$ the normalized arrhenius law

$$(3) \quad f_\varepsilon(u) := \frac{1}{\varepsilon^2} \exp\left(\frac{u - 1}{\varepsilon}\right),$$

where ε denotes the inverse of the (normalized) activation temperature. We also introduce μ , $0 < \mu < +\infty$, defined as

$$\mu := \lim_{\varepsilon \rightarrow 0} \int_{\theta_i}^1 f_\varepsilon(s)(1 - s)ds.$$

We assume that vaporisation starts at temperature $\theta_v > 0$, i.e. $\phi(u, m) = 0$ for all $0 \leq u < \theta_v$ and we also impose $\phi(u, 0) = 0$ for all $0 \leq u \leq 1$. It is also natural to assume that $\theta_v < \theta_i$ and to impose $\phi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$ is an increasing function of u , a decreasing function of m , and is a positive Lipschitz function on $[\theta_v, 1] \times [0, 1]$. Let be given any temperature θ , with $1 > \theta > \theta_v$, and any mass m . We finally make the natural assumption that the interval of time $\tau(\theta, m)$ needed for complete vaporisation of a droplet of initial mass m at temperature θ is finite.

1.2. Main results and summary. Section 2 is devoted to the proof of the existence of travelling waves solutions to system (2). Precisely,

Theorem 1. (existence of travelling waves) *System (2) admits a solution in $X = \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^0(\mathbb{R}) \times \mathbb{R}$.*

This result involves *a priori* bounds that are proved in Section 2.1, and where we emphasize the technical difficulties induced by the presence of vaporising droplets. This allows us to prove in Section 2.2 the existence of solutions to a system similar to (2) but defined on a bounded domain with appropriate boundary conditions. The proof of Theorem 1 follows in Section 2.3.

Section 3 is devoted to the asymptotic analysis of the system in the high activation energy limit. In Section 3.1, we prove the convergence of the system towards a Dirac model for spray flames:

Theorem 2. (the Dirac model for spray flames) *Let the reaction term $f_\varepsilon(\cdot)$ as in (3). There exists a decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $(u_{\varepsilon_n}, v_{\varepsilon_n}, m_{\varepsilon_n}, c_{\varepsilon_n})$ solution to (2) converges in $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times \mathbb{R}$ to (u, v, m, c) , solution to the problem*

$$(4) \quad \begin{aligned} -u'' + cu' &= c\delta_{x=\bar{x}}, \\ -\Lambda v'' + cv' &= -c\delta_{x=\bar{x}} - cn_0 m' \quad \text{on } \mathbb{R}, \\ cm' &= -\phi(u, m) \quad \text{on } \mathbb{R}, \\ u(-\infty) = 0, \quad u(0) = \theta, \quad u(+\infty) = 1, \\ v(-\infty) = v_u, \quad v(+\infty) = 0, \\ m(-\infty) = m_u. \end{aligned}$$

Here $\bar{x} = -\log \theta_i / c$.

For any given $c > 0$, the corresponding solution (u, v, m) is uniquely determined and one can find explicit expressions for all the unknowns, provided the expression for ϕ can be explicitly integrated. The complete characterisation of the system in the HAE limit is therefore dictated by the exact value of the velocity $c > 0$ of the travelling wave. The latter is determined thanks to the analysis of the internal combustion layer, also proved in Section 3.1:

Theorem 3. (internal combustion layer analysis) *Let (u, v, m, c) the limiting profile in the high activation energy limit. The velocity c is given by*

$$c = \min \left\{ \sqrt{2\mu/\Lambda}, c_\star(m_u) \right\},$$

where $c_\star(m_u)$ is the unique velocity such that the droplets finish vaporising exactly at the position of the reaction zone.

This result provides a rigorous justification of the existence of a vaporisation controlled regime, i.e. a regime where $c = c_\star(m_u) < \sqrt{2\mu/\Lambda}$ (see [38]). This regime appears when the time for complete vaporisation of the droplets is large enough, precisely when $m_u > m_u^\star$, where m_u^\star is defined by $c_\star(m_u^\star) = \sqrt{2\mu/\Lambda}$.

Section 3.2 provides more details about the internal combustion layer in the vaporisation controlled regime. An upper estimate of the size of the region where combustion and vaporisation overlap is given in Theorem 4, followed by an application to the so-called d^2 -law for vaporisation.

Finally, Section 4 presents some immediate applications of our analysis to the case of polydisperse sprays, fast vaporisation, or radial geometry. We refer to [6] for a more detailed exposition of those results.

Sketch of typical profiles. The pictures below sketch typical profiles for spray travelling waves moving from the right to the left. The temperature profile u is in red, the gaseous reactant profile v is in blue and the liquid phase density profile $n_0 m$ is in green. Recall that $u(-\infty) = 0$, $v(-\infty) = v_u$, $m(-\infty) = m_u$ and $u(+\infty) = 1 = v_u + n_0 m_u$. We set $n_0 = 1$.

Figure 1 shows typical profiles for gas flames in the absence of droplets, for a Lewis number equal to unity. In Figure 2, droplets are present in the fresh gas with mass density $n_0 m_u = 0.4$. We therefore set $v_u = 0.6$ so that the temperature in the burnt gas remains equal to unity. Notice that the gaseous reactant profile is no longer monotoneous, due to the vaporisation of the liquid phase.

Figures 3 & 4 are typical profiles in the high activation energy limit. The reaction zone is reduced to a point located at $\bar{x} = 0$ in Figure 3 and at $\bar{x} = 2.5$ in Figure 4. In both cases vaporisation starts at $x_v = -7.5$. However, in the first case the vaporisation front, i.e. the point x_{vf} where vaporisation ends, is located **before** the reaction front, at $x_v = -2.5 < 0$. Our analysis will show that in such a situation, the velocity of the spray flame in the HAE limit is equal to that of the purely gaseous flames with same temperature in the burnt gas. In the second case, the vaporisation ends at $x_{vf} = 2.5$. Shall the temperature profile remain the same as in Figure 3, the vaporisation process would still occur **after** the reaction zone. Our analysis will show that this is an **impossibility** and that the preheating zone “has to” stretch so that the reaction front $\bar{x} = 0$ from Figure 3 is “pushed” to the right until it coincides with the vaporisation front $\bar{x} = x_{vf} = 2.5$ as in Figure 4. The stretching of the preheating zone is synonymous with a decrease of the velocity of the flame. This is the so-called vaporisation controlled regime (see [38]).

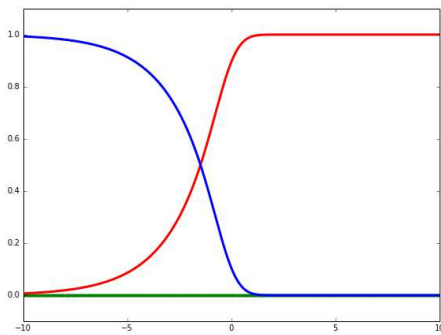


FIGURE 1

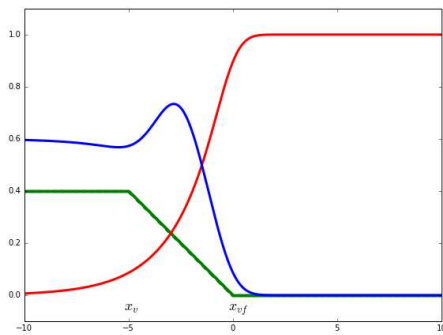


FIGURE 2

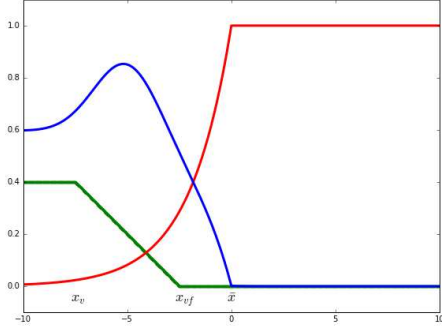


FIGURE 3

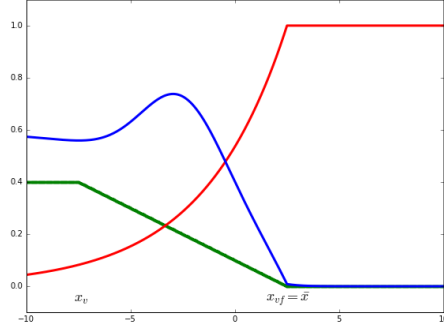


FIGURE 4

2. EXISTENCE OF TRAVELLING WAVES

Let $[-a, a]$ an interval of \mathbb{R} for some $a > 0$. Let $X_a = \mathcal{C}^1([-a, a]) \times \mathcal{C}^1([-a, a]) \times \mathcal{C}^0([-a, a]) \times \mathbb{R}$. We consider system (2) restricted to the interval $[-a, a]$ with Dirichlet boundary conditions at $x = a$ and flux conditions at $x = -a$,

$$\begin{aligned}
 (5) \quad & \begin{aligned}
 -u'' + cu' &= f(u)v && \text{on } (-a, a), \\
 -\Lambda v'' + cv' &= -f(u)v - cn_0 m' && \text{on } (-a, a), \\
 cm' &= -\phi(u, m) && \text{on } (-a, a), \\
 -u'(-a) + cu(-a) &= 0, && u(a) = 1, \\
 -\Lambda v'(-a) + cv(-a) &= cv_u, && v(a) = 0, \\
 m(-a) &= m_u, \\
 u(0) &= \theta_i.
 \end{aligned}
 \end{aligned}$$

The last equation allows one to break the translation invariance in the limit where a goes to infinity. We first prove *a priori* estimates for solutions $(u, v, m, c) \in X_a$. This allows us to establish the existence of solutions in a bounded domain by using topological degree arguments. Finally, the existence is extended to the real line.

2.1. Preliminary estimates. The main result of this section consists of *a priori* estimates for solutions to problem (5) above that are uniform with respect to the size a of the domain. We prove successively the following three results:

Proposition 1. (qualitative properties) *Let $(u, v, m, c) \in X_a$ a solution to (5) with $c \geq 0$. We have,*

$$\begin{aligned}
 (6) \quad & c > 0, \\
 (7) \quad & 0 < u \leq 1, \quad 0 \leq v < 1, \quad 0 \leq m \leq m_u \quad \text{on } [-a, a], \\
 (8) \quad & 0 < u' \leq c, \quad -c \leq \Lambda v' \leq c(1 + n_0 m_u), \quad m' \leq 0, \quad \text{on } [-a, a].
 \end{aligned}$$

Proposition 2. (a priori estimates) *Let (u, v, m, c) a solution to (5) with $c \geq 0$. We have*

$$\beta_1(\Lambda^{-1})(1 - u(x)) - n_0 m(x) \leq v(x) \leq \beta_2(\Lambda)(1 - u(x)),$$

where $\beta_1(\Lambda) := \min(1, \Lambda^{-1})$ and $\beta_2(\Lambda) := \max(1, \Lambda^{-1})$.

In the following we note, for all $\theta_i \leq s \leq 1$,

$$G(s) = \int_{\theta_i}^s f(s)(1-s)ds.$$

Proposition 3. (bounds on the velocity) *Let $\theta^* := (\theta_v + \theta_i)/2$, $c_* = \sqrt{\tau(m_u, \theta^*)/\log(\theta_i/\theta^*)}$ and $a_* = \log(\theta_i/\theta_v)/c_*$. For all $a \geq a_*$, the solution (u, v, m, c) to problem (5) with $c \geq 0$ obeys*

$$(9) \quad \min(c_*, c_1) \leq c \leq (c_2(a), c_3),$$

where we introduced $c_1 = \sqrt{2\beta_1(\Lambda)G(1)}/\theta_i$, $c_2(a) = \sqrt{2\beta_2(\Lambda)(G(1) + u'(a)^2)}/\theta_i$, $M_f = \sup_{s \in [0,1]} f(s)$, $c_3 = \max\left(M_f, \log(4/\theta_i)/a_*, 2\sqrt{M_f/\theta_i}\right)$.

Proof of Proposition 1 (qualitative properties) The presence of vaporising fuel in the model does not allow one to recover all the qualitative properties found in the gaseous case. For example, an important difference is the non-monotonicity of the reactant profile $v(x)$. Also, we introduce below auxiliary functions (such as v_* , \bar{w} or \bar{v}) that are specific to the analysis of spray flames, and specific steps such as the proof that $v \geq 0$ or the localisation of the vaporisation region.

Proof that $c > 0$. Recall that we assumed $c \geq 0$. Assume by contradiction that $c = 0$. Then the vaporisation term cn_0m' disappears in the equation of the reactant v . The function $y := u + \Lambda v - 1$ therefore solves $y'' = 0$ on $(-a, a)$, $y(-a) = y(a) = 0$, hence $y = 0$ on $[-a, a]$. It follows that (u, c) solves the nonlinear Dirichlet–Neumann boundary value problem

$$\begin{aligned} -u'' - \frac{1}{\Lambda}f(u)(1-u) &= 0 \quad \text{on } (-a, a), \\ u'(-a) &= 0, \quad u(0) = \theta, \quad u(a) = 1. \end{aligned}$$

We then observe that $u \leq 1$ on $[-a, a]$ otherwise the maximum $u(x_0) > 1 > \theta$ of u would be attained at $x_0 \in [-a, a]$ with $f(u(x_0)) > 0$. The equation above would imply $u''(x_0) > 0$ in contradiction with x_0 being a maximum. It follows that $w := u'$ obeys $w' \leq 0$ and $w(-a) = 0$, hence $w \leq 0$ and u decreasing on $[-a, a]$; a contradiction with $u(0) = \theta < 1 = u(a)$.

Proof that $v \geq 0$. Given a solution $(u, v, m, c) \in X_a$, introduce v_* the auxiliary reactant profile solution to the linear elliptic boundary value problem

$$\begin{aligned} \mathcal{L}v_* &:= -\Lambda v_*'' + cv_*' - f(u)v_* = 0 \quad \text{on } (-a, a), \\ \mathcal{N}v_*(-a) &:= -\Lambda v_*'(-a) + cv_*(-a) = 0, \quad v_*(a) = 0. \end{aligned}$$

The positive function $\phi \equiv 1$ obeys $\mathcal{L}\phi \geq 0$ on $(-a, a)$, $\mathcal{N}\phi(-a) = c > 0$. Therefore the problem has a unique solution, obviously $v_* \equiv 0$, and the generalized maximum

principle implies, since $\mathcal{L}v = -cn_0m' \geq 0$, $\mathcal{N}v(-a) = cv_u \geq 0$ and $v(a) \geq 0$, that $v \geq v_* \geq 0$ (see e.g. [33, Chapter 2]).

Bounds for u , v and m . The bound $0 \leq m(x) \leq m_u$ is trivial. Let $w := -\Lambda v' + cv + cn_0m$. We have $w' = -f(u)v \leq 0$, therefore $c = w(-a) \geq w(x) \geq w(a) = -\Lambda v'(a) + cn_0m(a) \geq 0$. We used $v'(a) \leq 0$, a consequence of $v \geq 0$ and $v(a) = 0$. Integrating the equation $-u'' + cu' = -w'$ on $[-a, x]$ yields $-u'(x) + cu(x) = -e^{cx}(e^{-cx}u)' = c - w(x)$. Therefore $-ce^{-cx} \leq (e^{-cx}u)' \leq 0$ and integrating now on $[x, a]$ yields exactly $0 \leq u(x) \leq 1$.

In order to bound v from above, we introduce the auxiliary function \bar{v} , solving

$$(10) \quad \begin{aligned} -\Lambda \bar{v}'' + c\bar{v}' &= -f(u)v \quad \text{on } (-a, a), \\ -\Lambda \bar{v}'(-a) + c\bar{v}(-a) &= c, \quad \bar{v}(a) = 0. \end{aligned}$$

Then let $\bar{w} := \bar{v} - v$, solution to $-\Lambda \bar{w}'' + c\bar{w}' = cn_0m'$ on $(-a, a)$, $-\Lambda \bar{w}'(-a) + c\bar{w}(-a) = cn_0m_u$ and $w(a) = 0$. Integrating over $[-a, x]$ yields $-\Lambda e^{cx/\Lambda}(e^{-cx/\Lambda}\bar{w})' = cn_0m(x)$, and integrating now on $[x, a]$ yields

$$(11) \quad \Lambda \bar{w}(x) = cn_0 \int_x^a e^{-c(s-x)/\Lambda} m(s) ds.$$

obviously $\bar{w} \geq 0$, that is $v \leq \bar{v}$ and it remains to bound \bar{v} from above. For that we observe that $-\Lambda \bar{v}'' + c\bar{v}' = +w'$. Integrating on $[-a, x]$ yields $(-c/\Lambda)e^{-cx/\Lambda} \leq (e^{-cx/\Lambda}\bar{v})' \leq 0$, and integrating now on $[x, a]$ yields exactly $0 \leq \bar{v} \leq 1$.

Bounds for u' and v' . We have from the previous step $u'(x) = c(u(x) - 1) + w(x) \leq w(x) \leq c$ since $u \leq 1$ and $w \leq c$. Also the equation for u rewrites $-e^{cx}(e^{-cx}u')' = f(u)v \geq 0$ and integrating from x to a yields $e^{-ca}u'(a) \leq e^{-cx}u'(x)$. Integrating the equation $-(u + \Lambda \bar{v})'' + c(u + \bar{v})' = 0$ on $(-a, a)$ allows us to estimate $u'(a) = -\Lambda \bar{v}'(a) > 0$, since $\bar{v} \geq v \geq 0$, $\bar{v}(a) = 0$ and \bar{v} is not identically zero. The monotonicity $u' > 0$ follows. Let again $w := -\Lambda v' + cv + cn_0m$. The bounds on $0 \leq w \leq c$, $0 \leq v \leq 1$ and $0 \leq m \leq m_u$ easily imply $-c \leq v' \leq c(1 + n_0m_u)$.

This concludes the proof of Proposition 1 (**qualitative properties**). \square

Proof of Proposition 2 (*a priori* estimates) In order to obtain a pointwise comparison of v with $(1 - u)$, we take a detour and compare rather \bar{v} with $(1 - u)$, where $0 \leq \bar{v} \leq 1$ is the auxiliary function defined in (10). Observe first that \bar{v} is nonincreasing. Indeed from (10) we have $(e^{-cx/\Lambda}\bar{v}')' \geq 0$, and integrating on $[x, a]$ yields $e^{-cx/\Lambda}\bar{v}'(x) \leq e^{-ca/\Lambda}\bar{v}'(a) \leq 0$.

Following [4], introduce now the two auxiliary functions $z := u + \bar{v} - 1$ and $y := u + \Lambda \bar{v} - 1$ and the four relations

$$\begin{aligned} -z'' + cz' &= (\Lambda - 1)\bar{v}'', \\ -\Lambda z'' + cz' &= (1 - \Lambda)u'', \\ -y'' + cy' &= (\Lambda - 1)c\bar{v}', \\ -\Lambda y'' + cy' &= (1 - \Lambda)c\bar{u}'. \end{aligned}$$

Integrating these equations successively on $[-a, x]$ and $[x, a]$ yields

$$\begin{aligned} z(x) &= (\Lambda - 1) \int_x^a e^{-c(s-x)} \bar{v}'(s) ds, \\ z(x) &= \frac{1 - \Lambda}{\Lambda} \int_x^a e^{-c(s-x)/\Lambda} u'(s) ds, \\ y(x) &= (1 - \Lambda)c \int_x^a e^{-c(s-x)} \bar{v}(s) ds, \\ y(x) &= \frac{\Lambda - 1}{\Lambda} c \int_x^a e^{-c(s-x)/\Lambda} (1 - u)(s) ds. \end{aligned}$$

Since \bar{v} and $(1 - u)$ are nonincreasing, it is not difficult to obtain

$$\begin{aligned} |z(x)| &\leq |\Lambda - 1| \bar{v}(x), & |z(x)| &\leq \left| \frac{\Lambda - 1}{\Lambda} \right| (1 - u(x)), \\ |y(x)| &\leq |\Lambda - 1| \bar{v}(x), & |y(x)| &\leq |\Lambda - 1| (1 - u(x)), \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{\Lambda} (1 - u(x)) &\leq \bar{v}(x) \leq (1 - u(x)) && \text{if } \Lambda > 1 \\ (1 - u(x)) &\leq \bar{v}(x) \leq \frac{1}{\Lambda} (1 - u(x)) && \text{if } 0 < \Lambda < 1. \end{aligned}$$

It remains to compare precisely \bar{v} and $v = \bar{v} - \bar{w}$. But since m is nonincreasing nonnegative, the expression (11) ensures pointwise $0 \leq \bar{w}(x) \leq n_0 m(x)$. The result announced follows.

This concludes the proof of Proposition 2 (***a priori* estimates**) . \square

Proof of Proposition 3 (bounds on the velocity) The nonlinear eigenvalue c is estimated thanks to energy estimates.

Estimates on $[0, a]$. Integrating successively the equation $f(u)v = -u'' + cu'$ against 1, u and u' on $[0, a]$ yields

$$\int_0^a f(u)v \cdot 1 = c - u'(a), \quad (\star)$$

$$\int_0^a f(u)v \cdot u = -u'(a) + \frac{c}{2}(1 + \theta_i)^2 + \int_0^a (u')^2, \quad (\star\star)$$

$$\int_0^a f(u)v \cdot u' = -\frac{1}{2}u'(a)^2 + \frac{1}{2}c^2\theta_i^2 + c \int_0^a (u')^2. \quad (\star\star\star)$$

where we used $u(0) = \theta_i$, $u'(0) = c\theta_i$. The combination $(\star\star\star) + c[(\star) - (\star\star)]$ reads

$$\begin{aligned} (\star\star\star) &:= \int_0^a f(u)v \cdot u' + c \int_0^a f(u)v \cdot (1 - u) \\ &= -\frac{1}{2}u'(a)^2 + \frac{c^2}{2}\theta_i^2. \end{aligned}$$

Since $(1 - u) \geq 0$, the two equations $(\star\star\star)$ and $(\star\star\star\star)$ imply

$$\int_0^a f(u)v \cdot u' \leq \frac{1}{2}c^2\theta_i^2 \leq \int_0^a f(u)v \cdot u' + \frac{1}{2}u'(a)^2.$$

It remains to estimate the contribution of the reaction term in the inequalities above.

First upper bound for c . From Proposition 2, we have $v(x) \leq \beta_2(\Lambda)(1 - u(x))$ on $[0, a]$, hence

$$\int_0^a f(u)v \cdot u' \leq \beta_2(\Lambda) \int_0^a f(u(x))(1 - u(x)) \cdot u'(x)dx = \beta_2(\Lambda)G(1),$$

which provides the value of $c_2(a)$ in Proposition 3 such that

$$\frac{1}{2}c_2(a)^2\theta_i := \beta_2(\Lambda)G(1) + \frac{1}{2}u'(a)^2.$$

Localisation of the vaporisation region. We claim that for small enough velocities, the vaporisation region does not intersect the combustion zone. Recall that $\theta_v < \theta_i$ so that $\theta^* := (\theta_v + \theta_i)/2 < \theta_i$. Let $\tau(m_u, \theta^*)$ the interval of time needed for complete vaporisation of a droplet of size m_u at constant temperature θ^* . From the hypothesis on the vaporisation term, we have $\tau(m_u, \theta^*) < \infty$. Given $c > 0$, let x_v such that $u(x_v) = \theta_v$, x^* such that $u(x^*) = \theta^*$ and $x_i = 0$ such that $u(x_i) = \theta_i$. Since $u(x) = \theta_i e^{cx}$ on $[-a, 0]$, we have for any given c and a large enough, $-a \leq x_v < x^* < x_i$ and $x_i - x^* = \log(\theta_i/\theta^*)/c$. The time τ^* spent by the droplets advected at velocity c inside the interval $[x^*, x_i]$ is $\tau^* = (x_i - x^*)/c = \log(\theta_i/\theta^*)/c^2$. However, since $m(x^*) \leq m_u$, and $u \geq \theta^*$ on $[x^*, x_i]$, the monotonicity properties of the vaporisation law imply that

if $\tau^* \geq \tau(m_u, \theta^*)$ hence complete vaporisation occurs inside the interval $[x^*, x_i]$. To prove our claim, it suffices to take $a \geq a_*$ and $c \leq c_*$, with

$$c_* = \sqrt{\tau(m_u, \theta^*) / \log(\theta_i / \theta^*)}, \quad a_* = \log(\theta_i / \theta_v) / c_*.$$

Lower bound for c. Take $a \geq a_*$. If $c \geq c_*$ there is nothing to prove. Otherwise, we have $c \leq c_*$, $m(x) = 0$ on $[0, a]$ so that from Proposition 2 we have $v(x) \geq \beta_1(\Lambda)(1 - u(x))$, therefore

$$\int_0^a f(u)v \cdot u' \geq \beta_1(\Lambda) \int_0^a f(u(x))(1 - u(x)) \cdot u'(x) dx = \beta_1(\Lambda)G(1),$$

which allows us to define the value c_1 in Proposition 3 thanks to the relation $c_1^2 \theta_i / 2 := \beta_1(\Lambda)G(1)$.

Second upper bound for c. Let a_* defined above. Let $M := \sup_{s \in [0,1]} f(s)$. Let \bar{u} be the solution to $-\bar{u}'' + c\bar{u}' = M\mathbb{1}_{]0,a[}$ on $] -a, a[$, $-\bar{u}'(-a) + c\bar{u}(-a) = 0$ and $\bar{u}(a) = 1$. Since $0 \leq v \leq 1$, the maximum principle asserts that $u \leq \bar{u}$ on $[-a, a]$, in particular $\theta_i \leq \bar{\theta} := \bar{u}(0)$. Solving explicitly for \bar{u} yields

$$\bar{u}(0) = e^{-ca}(1 - M/c) + (1 - e^{-ca})M/c^2.$$

Assume $a \geq a_* > 0$, then the right hand side tends towards zero as c goes to infinity. More precisely, if both $c \geq M$, $e^{-ca_*} \leq \theta_i/4$ and $M/c^2 \leq \theta_i/4$ one has $\bar{u}(0) = \bar{\theta} \leq \theta_i/2$, a contradiction. It follows that

$$c_3 := \max \left(M, \log(4/\theta_i)/a_*, 2\sqrt{M/\theta_i} \right)$$

is an upper-bound for c .

This concludes the proof of Proposition 3 (**bounds on the velocity**) . \square

2.2. Existence of a solution on bounded domains. The proof of the existence of a topological degree relies on the existence of the continuous map K_τ below that is not continuous if the reaction or vaporisation terms have discontinuities at θ_i or θ_v respectively. Following BERESTYCKI–NICOLAENKO–SCHEURER (see [4, footnote p. 1225]), we assume below that f and g are continuous. The case where a discontinuity is present can be attained by a standard smoothing procedure.

The main point when dealing with a spray flame model is to correctly account for the vaporisation terms in the homotopy argument in order to preserve the *a priori* estimates of the previous section.

Proposition 4. (existence of a solution on bounded domains) *There exists $a_0 > 0$ such that for $a \geq a_0$ system (5) admits a solution in $X_a = \mathcal{C}^1([-a, a]) \times \mathcal{C}^1([-a, a]) \times \mathcal{C}^0([-a, a]) \times \mathbb{R}$.*

Proof. We use the LERAY–SCHAUDER topological degree argument. Let $\tau \in [0, 1]$ the homotopy parameter and consider the solutions $(u_\tau, v_\tau, m_\tau, c_\tau)$ to the new system

$$\begin{aligned}
 (12) \quad & -u'' + cu' = \tau\{f(u)v\} \text{ on } (-a, a), \\
 & -\Lambda v'' + cv' = \tau\{-f(u)v + n_0\phi(u, m)\} \text{ on } (-a, a), \\
 & cm' = \tau\{-\phi(u, m)\} \text{ on } (-a, a), \\
 & -u'(-a) + cu(-a) = 0, \quad u(a) = 1, \\
 & -\Lambda v'(-a) + cv(-a) = cv_u, \quad v(a) = 0, \\
 & m(-a) = m_u, \\
 & c = u(0) - \theta_i + \tau c.
 \end{aligned}$$

This is system (5) where the condition $u(0) = \theta_i$ has been rewritten as a fixed point for the velocity c and where the homotopy parameter appears in the right-hand-side of the system. It is important to notice that this system can be obtained by replacing the vaporisation terms $f(u)$ and $\phi(u, m)$ in (5) by $f_\tau := \tau f$ and $\phi_\tau := \tau \phi$. It follows that all *a priori* estimates from the previous sections hold.

Fixed point formulation. We look for solutions $(u_\tau, v_\tau, m_\tau, c_\tau)$ defined as a fixed point of the mapping $K_\tau : X_a \rightarrow X_a$ that maps (u, v, m, c) to $(\hat{u}, \hat{v}, \hat{m}, \hat{c})$ solution to the linear boundary value problem

$$\begin{aligned}
 (13) \quad & -\hat{u}'' + c\hat{u}' = \tau\{f(u)v\} \text{ on } (-a, a), \\
 & -\Lambda \hat{v}'' + c\hat{v}' = \tau\{-f(u)v + n_0\phi(u, m)\} \text{ on } (-a, a), \\
 & c\hat{m}' = \tau\{-\phi(u, m)\} \text{ on } (-a, a), \\
 & -\hat{u}'(-a) + c\hat{u}(-a) = 0, \quad \hat{u}(a) = 1, \\
 & -\Lambda \hat{v}'(-a) + c\hat{v}(-a) = cv_u, \quad \hat{v}(a) = 0, \\
 & \hat{m}(-a) = m_u, \\
 & \hat{c} = u(0) - \theta_i + \tau c.
 \end{aligned}$$

Since $H^2([-a, a])$ embeds compactly in $C^1([-a, a])$ and $H^1([-a, a])$ embeds compactly in $C([-a, a])$, it follows that K_τ is a compact mapping and uniformly continuous with respect to τ . Let $F_\tau := \text{Id} - K_\tau$. A solution $(u_\tau, v_\tau, m_\tau, c_\tau)$ to system (12) is a fixed point of F_τ , i.e. $F_\tau(u_\tau, v_\tau, m_\tau, c_\tau) = 0$.

Existence of degree of F_τ . A solution to (12) for $0 \leq \tau \leq 1$ exists as soon as the degree F_τ is well defined and non zero. Let $\Omega \subset X_a$ the open set

$$\Omega = \{(u, v, m, c); \|u\|_{C^1(\bar{I}_a)} \leq M, \|v\|_{C^1(\bar{I}_a)} \leq M, \|m\|_{C(\bar{I}_a)} \leq M, \underline{c} < c < \bar{c}\}$$

for some positive constants $M, \underline{c}, \bar{c}$. These constants can be chosen so that for all $0 \leq \tau \leq 1$, $F_\tau \neq 0$ on $\partial\Omega$. Indeed, Proposition 2 provides estimates for the case $\tau = 1$, namely $0 < c_\star/2 < c < 2c^\star$, $\|u\|_{C^1(\bar{I}_a)} \leq 1 + c \leq 1 + c^\star$, $\|v\|_{C^1(\bar{I}_a)} \leq 1 + \beta_2(\Lambda)c^\star(1 + n_0m_u)$, $\|m\|_{C(\bar{I}_a)} \leq m_u$. Setting $M := \max(1 + c^\star, 1 + \beta_2(\Lambda)c^\star(1 + n_0m_u), m_u)$, $\underline{c} := c_\star/2$, $\bar{c} := 2c^\star$ ensures $F_1(u, v, m, c) \neq 0$ for any $(u, v, m, c) \in \partial\Omega$. Notice now that rescaling $f_\tau := \tau f$ and $\phi_\tau = \tau \phi$ in (12), leads to a fixed point problem similar to F_1 but with rescaled reaction and vaporisation terms. The conclusions of Proposition 2 still hold

with the same bounds, ensuring that for all $0 \leq \tau \leq 1$, $F_\tau(u, v, m, c) \neq 0$ for any $(u, v, m, c) \subset \partial\Omega$.

Calculation of the degree of F_τ . Thanks to the properties of K_τ , and the homotopy invariance of the degree we have $\deg(F_\tau, \Omega, 0) = \deg(F_0, \Omega, 0)$ for all $0 \leq \tau \leq 1$ and it suffices to compute the degree of F_0 . Given (u, v, m, c) , the solution $(\hat{u}, \hat{v}, \hat{m}, \hat{c})$ to (13) with $\tau = 0$ is

$$\begin{aligned}\hat{u}(c) &= e^{c(x-a)}, \quad \hat{v}(c) = v_u(1 - e^{c(x-a)/\Lambda}), \\ \hat{m}(u, m, c) &= \mathcal{M}(u, m, c), \quad \hat{c}(u, c) = c - u(0) + \theta_i.\end{aligned}$$

The mapping

$$F_0 : (u, v, m, c) \rightarrow (u - \hat{u}(c), v - \hat{v}(c), m - \hat{m}(u, m, c), u(0) - \theta_i)$$

is homotopic to

$$\tilde{F}_0 : (u, v, m, c) \rightarrow (u - \hat{u}(c), v - \hat{v}(c), m - \hat{m}(u, m, c), \hat{u}(0) - \theta_i).$$

This linear system has a unique solution in X_a therefore its degree is different from 0. The same result holds for F_1 which therefore admits at least one solution in X_a . \square

2.3. Existence of travelling waves. We are now able to prove the first main result of the paper, that is the existence of a travelling wave for system (2) on the real line.

Proof of Theorem 1 Let (u_a, v_a, m_a, c_a) the solution to (5) for $a \geq a_\star$. We proved that (u_a, v_a, m_a, c_a) is bounded in $W^{1,\infty}(-a, a) \times W^{1,\infty}(-a, a) \times L^\infty(-a, a) \times \mathbb{R}$ uniformly with respect to $a \geq a_\star$. Using system (5), the uniform boundedness also holds in $W^{2,\infty}(-a, a) \times W^{2,\infty}(-a, a) \times L^\infty(-a, a) \times \mathbb{R}$. Let $(a_n)_{n \geq 0}$, with $a_n \geq a_\star$ an increasing sequence tending to infinity. We can extract a sequence $(u_{a_n}, v_{a_n}, m_{a_n}, c_{a_n})$ converging in $\mathcal{C}_{\text{loc}}^1(\mathbb{R}) \times \mathcal{C}_{\text{loc}}^1(\mathbb{R}) \times \mathcal{C}_{\text{loc}}^0(\mathbb{R}) \times \mathbb{R}$ solution to

$$\begin{aligned}-u'' + cu' &= f(u)v \quad \text{on } \mathbb{R}, \\ -\Lambda v'' + cv' &= f(u)v - cn_0 m' \quad \text{on } \mathbb{R}, \\ cm' &= -\phi(u, m) \quad \text{on } \mathbb{R}, \\ u(0) &= \theta_i.\end{aligned}$$

From the explicit expressions of u_{a_n} and v_{a_n} on $(-a_n, 0)$ it follows that $u(-\infty) = 0$ and $v(-\infty) = v_u$. Moreover, since the velocity c_{a_n} is uniformly bounded below by a certain $\underline{c} > 0$, the vaporisation point $x_v^{a_n}$ such that $u(x_v^{a_n}) = \theta_v$ is bounded below by $x_v^\star = -\log(\theta_i/\theta_v)/\underline{c}$ and $m_{a_n} = m_u$ on $(-a_n, x_v^\star)$. It follows $m(-\infty) = m_u$. Now, v and v' inherit the uniform bounds of v_{a_n} and v'_{a_n} , so that $v'(+\infty)$ is bounded, therefore $v'(+\infty) = 0$ since v is bounded. From the equation satisfied by v , since $\phi(u, m)$ and $f(u)v$ are bounded, we have also that v'' is bounded, therefore $v''(+\infty) = 0$. Let $x_{vf}^{a_n}$ the vaporisation front, that is the first point where $m_{a_n}(x_{vf}^{a_n}) = 0$. Since c_{a_n} is bounded above by some $\bar{c} > 0$, and since by hypothesis the droplets vaporise in finite

time, one can bound uniformly $x_{vf}^{a_n} < x_{vf}^*$ for some $x_{vf}^* > 0$, and for all a_n , $m_{a_n}(x) = 0$ if $x \geq x_{vf}^*$. It follows $m(\infty) = 0$, and the equation for v implies $f(u(+\infty))v(+\infty) = 0$. The monotonicity of u holds for the same reason as the monotonicity of u_{a_n} holds, therefore $u(+\infty) > \theta_i$ and $f(u(+\infty)) > 0$, hence $v(+\infty) = 0$. Finally, since $m(x) = 0$ on $(x_{vf}, +\infty)$, we have from Proposition 2 that $v \geq \beta_1(\Lambda)(1 - u) \geq 0$ on $(x_{vf}, +\infty)$ hence $u(+\infty) = 1$. \square

3. HIGH ACTIVATION ENERGY LIMIT

3.1. Characterisation of the limiting profiles. In this section we assume a reaction term $f_\varepsilon(u)$ obeying the Arrhenius law (3) on $[\theta_i, 1]$. Set $\theta_\varepsilon := 1 + A\varepsilon \ln \varepsilon < 1$ with $A = 100$ and ε small enough. Notice $\theta_i < \theta_\varepsilon$ for ε small enough. We have immediately

$$(14) \quad \lim_{\varepsilon \rightarrow 0} \theta_\varepsilon = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \max_{\theta_i \leq s \leq \theta_\varepsilon} f_\varepsilon(s)(1 - s) = 0,$$

$$(15) \quad \lim_{\varepsilon \rightarrow 0} \int_{\theta_i}^1 f_\varepsilon(s)(1 - s)ds = \lim_{\varepsilon \rightarrow 0} G_\varepsilon(1) =: \mu < \infty.$$

We follow and adapt the high energy activation analyses and tools from [4, 9, 10].

Proof of Theorem 2 (limiting system in the HAE limit) Choose $\varepsilon > 0$ small enough so that $\mu/2 \leq G_\varepsilon(1) \leq 2\mu$. The bounds (9) for the velocity with $a = +\infty$ and $u'_\varepsilon(+\infty) = 0$ allow us to estimate independently of ε

$$\underline{c}/\sqrt{2} \leq c_\varepsilon \leq \sqrt{2}\bar{c},$$

with $\underline{c} := \min(c_*, c_1)$, c_* as in Proposition 3, $c_1 = \sqrt{2\beta_1(\Lambda)\mu}/\theta_i$, and $\bar{c} := \sqrt{2\beta_2(\Lambda)\mu}/\theta_i$.

We can therefore find a decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that c_{ε_n} converges

$$c_{\varepsilon_n} \rightarrow c > 0.$$

For ε small enough, let $x_\varepsilon > 0$ be the unique point satisfying $u_\varepsilon(x_\varepsilon) = 1 + A\varepsilon \ln \varepsilon =: \theta_\varepsilon < 1$, where A is a fixed sufficiently large constant (take $A = 100$). Let x_v^ε be the point where vaporisation starts, i.e. $u_\varepsilon(x_v^\varepsilon) = \theta_v$. Let $x_v^\varepsilon \leq x \leq x_\varepsilon$. Rewriting equation $(\star\star\star)$ on $[x_v^\varepsilon, x]$ and comparing v_ε with u_ε yields

$$\begin{aligned} \frac{1}{2}u'_\varepsilon(x)^2 &\geq \frac{1}{2}c_\varepsilon^2\theta_v^2 - \int_0^x f_\varepsilon(u_\varepsilon)v_\varepsilon \cdot u'_\varepsilon \\ &\geq \frac{1}{2}\underline{c}^2\theta_v^2 - \max(1, \Lambda) \int_0^x f_\varepsilon(u_\varepsilon)(1 - u_\varepsilon) \cdot u'_\varepsilon, \end{aligned}$$

where the last integral is $\mathcal{O}(\varepsilon)$ thanks to the properties of f_ε . Hence there exist constants $\alpha > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ and $x \in [x_v^\varepsilon, x_\varepsilon]$, we have $u'_\varepsilon(x) \geq \alpha$. Since $u_\varepsilon(0) = \theta_i < \theta_\varepsilon = u_\varepsilon(x_\varepsilon)$, it follows $0 < x_\varepsilon \leq (\theta_\varepsilon - \theta_v)/\alpha < (1 - \theta_v)/\alpha =: x_0 < +\infty$. The uniform boundedness of x_ε implies, up to extracting a subsequence,

$$x_{\varepsilon_n} \rightarrow \bar{x} \leq x_0.$$

Now for the point x_v^ε where vaporisation starts, we have $|x_v^\varepsilon - 0| \leq (\theta_v - \theta_i)/\alpha$, hence x_v^ε is bounded and converges, after extraction of a subsequence,

$$x_v^{\varepsilon_n} \rightarrow x_v.$$

Convergence of u_ε . The convergence of c_ε together with the explicit expressions of the exponential profile u_ε on $(-\infty, 0)$ implies the convergence in $H^1(-\infty, 0)$ of u_ε towards u . Inspecting the L^2 estimates in the proof of Proposition 3 yields the uniform $H^1(0, x_0)$ boundedness of u_ε , therefore convergence in $\mathcal{C}^0((0, x_0))$. Thanks to the monotonicity of u_ε , we have that $u_\varepsilon \rightarrow 1$ uniformly on $(\bar{x}, +\infty)$. Finally, the reaction term is uniformly bounded on $(-\infty, x_\varepsilon)$ by a multiple (14), which tends to 0 by hypothesis. This implies that the reaction term tends uniformly towards zero on any compact subset of $(-\infty, \bar{x})$. To summarize, u_ε converges in $H^1(\mathbb{R})$ towards a continuous profile u solution to $-u'' + cu' = 0$ on $(-\infty, \bar{x})$, $u = 1$ on $(\bar{x}, +\infty)$.

Convergence of m_ε . Since $\theta_v < 1$, we have for ε small enough that $x_v^\varepsilon < x_\varepsilon$, and $\theta_\varepsilon \geq (1 + \theta_v)/2 > \theta_v$. From the hypothesis on the vaporisation law, it follows that complete vaporisation occurs on a finite interval. Let x_{vf}^ε denote the unique position of the vaporisation front. The uniform boundedness of x_{vf}^ε implies the convergence

$$x_{vf}^{\varepsilon_n} \rightarrow x_{vf}.$$

Moreover, from the \mathcal{C}^0 convergence of u_ε towards u on the compact $[x_v - 1, x_{vf} + 1]$, together with the Lipschitz properties of the vaporisation terms, it follows that m_ε converges towards m on $[x_v - 1, x_{vf} + 1]$ in $\mathcal{C}^0([x_v - 1, x_{vf} + 1])$ solution to $cm' = -g(u, m)$. Since $m'_\varepsilon = 0$ outside of $[x_v^\varepsilon, x_{vf}^\varepsilon]$, the convergence holds on $\mathcal{C}^0(\mathbb{R})$ with m solving $m(-\infty) = m_u$ and $cm' = -g(u, m)$ on \mathbb{R} .

Convergence of v_ε . The convergence of v_ε follows the same line as that of u_ε and uses the convergence of m_ε . The limiting solution $v \in \mathcal{C}^0(\mathbb{R})$ satisfies $-\Lambda v'' + cv' = -cn_0 m'$ on $(-\infty, \bar{x})$, $v = 0$ on $(\bar{x}, +\infty)$.

Convergence on $(\bar{x}, +\infty)$. Since u_ε is increasing, we have for any $x_\varepsilon \leq x$, $u(x_{\varepsilon_n}) = \theta_{\varepsilon_n} \leq u_{\varepsilon_n}(x) \leq 1$ with $\theta_{\varepsilon_n} \rightarrow 1$. This implies easily $u_\varepsilon(x) \rightarrow 1$ on $(\bar{x}, +\infty)$. The upper bound $v_\varepsilon(x) \leq \beta_2(\Lambda)(1 - u_\varepsilon(x))$ ensures $v_\varepsilon(x) \rightarrow 0$ on $(\bar{x}, +\infty)$. Now, the auxiliary function $y_\varepsilon := u_\varepsilon + \Lambda v_\varepsilon - 1$ satisfies $y_\varepsilon(x_\varepsilon) = \theta_\varepsilon + \Lambda v_\varepsilon(x_\varepsilon) - 1$, $y_\varepsilon(+\infty) = 0$, and $-y_\varepsilon'' + c_\varepsilon y_\varepsilon' = (\Lambda - 1)c_\varepsilon v_\varepsilon' - \Lambda c_\varepsilon n_0 m'_\varepsilon$. Therefore the H^1 -limit y satisfies $y = u + \Lambda v - 1 = 0$ on $(\bar{x}, +\infty)$, together with $-y'' + cy' = (\Lambda - 1)cv' - \Lambda cn_0 m'$. This implies $m' = 0$ on $(\bar{x}, +\infty)$ therefore also $m(x) = 0$ on $(\bar{x}, +\infty)$. Obviously, u and v are \mathcal{C}^2 on $(\bar{x}, +\infty)$.

Estimate of x_{vf} . The fact that $m(x) = 0$ for $x \geq \bar{x}$ is equivalent to

$$x_{vf} \leq \bar{x},$$

that is in the HAE limit, the vaporisation region ends before or at the reaction zone.

Convergence on $(-\infty, \bar{x})$. Since u_ε is increasing and $u_\varepsilon(0) = \theta_i$, the reaction term is zero on $(-\infty, 0)$. From property (14) together with the upper estimate $v_\varepsilon \leq \beta_2(\Lambda)(1 - u_\varepsilon)$, we deduce that $f_\varepsilon(u_\varepsilon)v_\varepsilon$ converges uniformly to zero on any compact set of $[0, \bar{x})$, therefore also on any compact set of $(-\infty, \bar{x})$. This implies that the limiting functions u , v and m satisfy $-u'' + cu' = 0$, $-\Lambda v'' + cv' = -cn_0m'$, and $cn_0m' = -g(u, m)$ on $(-\infty, \bar{x})$, with $m' \in L^2((-\infty, \bar{x}))$, $v \in H^2((-\infty, \bar{x})) \subset \mathcal{C}^1((-\infty, \bar{x}))$ and $u(x) = \theta_i e^{c(x-\bar{x})} \in \mathcal{C}^2((-\infty, \bar{x}))$.

This concludes the proof of Theorem 2 (**limiting system in the HAE limit**) \square

Proof of Theorem 3 (internal layer analysis) Set now $u_\varepsilon(0) = 1 + A\varepsilon \ln \varepsilon =: \theta_\varepsilon$ at $x = 0$. Let then $x_i^\varepsilon \leq 0$ the ignition point where $u_\varepsilon(x_i^\varepsilon) = \theta_i$. We have seen in the proof of Theorem 2 above that $|x_i^\varepsilon - 0|$ is bounded independently of ε . From property (14), we have therefore $f_\varepsilon(u_\varepsilon)v_\varepsilon = \mathcal{O}(\varepsilon^{A/2})$ uniformly on $(x_i^\varepsilon, 0)$ and is zero on $(-\infty, x_i^\varepsilon)$. Integrating the equation for u_ε between $-\infty$ and 0, this implies

$$-u'_\varepsilon(0) + c_\varepsilon \theta_\varepsilon = \mathcal{O}(\varepsilon^{A/2}),$$

where $\theta_\varepsilon = 1 + \mathcal{O}(\varepsilon \ln \varepsilon)$. It follows that

$$c_\varepsilon = u'_\varepsilon(0) + \mathcal{O}(\varepsilon \ln \varepsilon).$$

Rescaled system. Let $\xi := x/\varepsilon$.

$$\hat{u}_\varepsilon(\xi) = \frac{u_\varepsilon(x) - 1}{\varepsilon}, \quad \hat{v}_\varepsilon(\xi) = \frac{v_\varepsilon(x)}{\varepsilon}, \quad m_\varepsilon(\xi) = \hat{m}_\varepsilon(x),$$

together with the usual auxiliary functions

$$z_\varepsilon(x) = u_\varepsilon(x) + v_\varepsilon(x) - 1, \quad y_\varepsilon(x) = u_\varepsilon(x) + \Lambda v_\varepsilon(x) - 1,$$

$$\hat{z}_\varepsilon(\xi) = \frac{z_\varepsilon(x)}{\varepsilon}, \quad \hat{y}_\varepsilon(\xi) = \frac{y_\varepsilon(x)}{\varepsilon}.$$

These functions obey the system of equations:

$$\begin{aligned} -\hat{u}_\varepsilon'' + \varepsilon c_\varepsilon \hat{u}_\varepsilon' &= \hat{v}_\varepsilon \exp(\hat{u}_\varepsilon) = (\hat{z}_\varepsilon - \hat{u}_\varepsilon) \exp(\hat{u}_\varepsilon) = \frac{1}{\Lambda}(\hat{y}_\varepsilon - \hat{u}_\varepsilon) \exp(\hat{u}_\varepsilon), \\ \hat{u}_\varepsilon(0) &= A \ln \varepsilon, \quad \hat{u}_\varepsilon(+\infty) = 0, \quad \hat{u}_\varepsilon(-\infty) = -1/\varepsilon, \\ -\hat{v}_\varepsilon'' + \varepsilon c_\varepsilon \hat{v}_\varepsilon' &= -\exp(\hat{u}_\varepsilon) \hat{v}_\varepsilon - c_\varepsilon n_0 \hat{m}_\varepsilon', \quad \hat{v}_\varepsilon(+\infty) = 0, \quad \hat{v}_\varepsilon(-\infty) = v_u/\varepsilon, \\ c_\varepsilon n_0 \hat{m}_\varepsilon' &= -g(1 + \varepsilon \hat{u}_\varepsilon) \phi(\hat{m}_\varepsilon), \quad \hat{m}_\varepsilon(-\infty) = m_u, \\ -\hat{z}_\varepsilon'' + \varepsilon c_\varepsilon \hat{z}_\varepsilon' &= (\Lambda - 1) \hat{v}_\varepsilon'' - c_\varepsilon n_0 \hat{m}_\varepsilon', \\ \hat{z}_\varepsilon(+\infty) &= 0, \quad \hat{z}_\varepsilon(-\infty) = (v_u - 1)/\varepsilon, \\ -\Lambda \hat{y}_\varepsilon'' + \varepsilon c_\varepsilon \hat{y}_\varepsilon' &= \varepsilon c_\varepsilon (1 - \Lambda) \hat{u}_\varepsilon' - c_\varepsilon n_0 \hat{m}_\varepsilon', \\ \hat{y}_\varepsilon(+\infty) &= 0, \quad \hat{y}_\varepsilon(-\infty) = (\Lambda v_u - 1)/\varepsilon. \end{aligned}$$

Notice that we have $u'_\varepsilon(0) = \hat{u}_\varepsilon'(0)$, therefore also $c_\varepsilon = \hat{u}_\varepsilon'(0) + \mathcal{O}(\varepsilon \ln \varepsilon)$. Our goal is to estimate $\hat{u}_\varepsilon'(0)$ in the limit $\varepsilon \rightarrow 0$.

Approximate system. Let \tilde{u}_ε and \tilde{y}_ε the solution to the linear system

$$(16) \quad -\tilde{u}_\varepsilon'' = \frac{1}{\Lambda}(\tilde{y}_\varepsilon - \tilde{u}_\varepsilon) \exp(\hat{u}_\varepsilon),$$

$$(17) \quad -\Lambda \tilde{y}_\varepsilon'' = -c_\varepsilon n_0 \hat{m}_\varepsilon',$$

with the same boundary conditions as \hat{u}_ε and \hat{y}_ε , and where we omitted the terms involving the factors $\varepsilon c_\varepsilon$. Classical results in elliptic regularity ensure that $(\hat{u}_\varepsilon, \hat{y}_\varepsilon) - (\tilde{u}_\varepsilon, \tilde{y}_\varepsilon)$ is of order $\mathcal{O}(\varepsilon)$ on $[0, \infty)$ in \mathcal{C}^1 . Therefore $c_\varepsilon = \tilde{u}_\varepsilon'(0) + \mathcal{O}(\varepsilon \ln \varepsilon)$.

Case $x_{vf} < 0$. Let x_{vf}^ε the position of the vaporisation front such that $\hat{m}_\varepsilon = 0$ on $(x_{vf}^\varepsilon, +\infty)$. If $x_{vf} < 0$, then for ε small enough, $\hat{m}_\varepsilon = 0$ and $\tilde{y}_\varepsilon = 0$ on $(0, +\infty)$ and

$$\begin{aligned} -\tilde{u}_\varepsilon'' &= -\tilde{u}_\varepsilon \exp(\hat{u}_\varepsilon)/\Lambda = -\tilde{u}_\varepsilon \exp(\tilde{u}_\varepsilon)/\Lambda + \tilde{u}_\varepsilon \mathcal{O}(\varepsilon) \\ &= -\tilde{u}_\varepsilon \exp(\tilde{u}_\varepsilon)/\Lambda + \mathcal{O}(\varepsilon \ln \varepsilon). \end{aligned}$$

Omitting the last term in the equation above, it remains to study the solution to the new approximate system with the same boundary conditions

$$-\tilde{u}_\varepsilon'' = -\tilde{u}_\varepsilon \exp(\tilde{u}_\varepsilon)/\Lambda.$$

Integrating against \tilde{u}_ε' on $(0, +\infty)$ yields

$$\frac{\tilde{u}_\varepsilon'(0)^2}{2} = \frac{\mu}{\Lambda} \int_{A \ln \varepsilon}^0 -\sigma e^\sigma,$$

where the integral in the right hand side tends to one as ε goes to zero. Hence

$$c_\varepsilon = u_\varepsilon'(0) = \tilde{u}_\varepsilon'(0) + \mathcal{O}(\varepsilon \ln \varepsilon) = \sqrt{\frac{2\mu}{\Lambda}} + \mathcal{O}(\varepsilon \ln \varepsilon),$$

and the limiting velocity is $c = \sqrt{2\mu/\Lambda}$ when $x_{vf} = 0$.

Case $x_{vf} = 0$. In the case where the vaporisation front is located in the limit at the position $\bar{x} = 0$ of the reaction zone, the velocity c is imposed by the condition $m(x) = 0$. Indeed, we have in the limit $u(x) = e^{cx}$ for $x \leq 0$. Assume $0 < c_1 < c_2$. The corresponding profiles satisfy $u_1 > u_2$ on $(-\infty, 0)$ and the corresponding positions where vaporisation starts satisfy $x_v^1 < x_v^2$. The monotonicity properties of the vaporisation law imply $x_{vf}^1 < x_{vf}^2$, where those are the vaporisation fronts associated to the velocities c_1 and c_2 respectively. Consequently, the vaporisation front x_{vf} is an increasing function $x_{vf} := x_{vf}(c)$ of the velocity c . With similar arguments, it is also an increasing function of m_u . Moreover, since the vaporisation law ϕ is Lipschitz on $(\theta_v, 1)$, it follows that $x_{vf}(c)$ is continuous and we can define in a unique manner $c_\star(m_u)$ to be the velocity such that $x_{vf}(c_\star(m_u)) = 0$.

As a conclusion, when $x_{vf} = 0$, we have $c = c_*(m_u)$, whereas $c = \sqrt{2\mu/\Lambda}$ for $x_{vf} < 0$. The monotonicity of $x_{vf} : c \mapsto x_{vf}(c)$ implies

$$\begin{aligned} c_*(m_u) &\leq \sqrt{2\mu/\Lambda}, \\ x_{vf} = 0 &\Leftrightarrow c = c_*(m_u) < \sqrt{2\mu/\Lambda}, \\ x_{vf} < 0 &\Leftrightarrow c = \sqrt{2\mu/\Lambda}. \end{aligned}$$

This concludes the proof of Theorem 3. \square

3.2. Internal combustion layer in the vaporisation controlled regime. We are interested in the case where the droplets finish vaporising inside the combustion layer. In order to measure the quantity of droplets still present in the combustion region $[0, +\infty)$, it is convenient to introduce $M_\varepsilon(x)$ the following primitive of the mass of liquid

$$M_\varepsilon(x) := \int_x^{+\infty} m_\varepsilon(x) dx.$$

Theorem 4. (estimate of the overlapping region) *Let M_ε defined above. We have*

$$M_\varepsilon(0) = \mathcal{O}(\varepsilon \ln \varepsilon).$$

Proof. Consider the approximate system on the interval $(0, +\infty)$

$$(18) \quad -\ddot{u}_\varepsilon = \frac{1}{\Lambda}(\check{y}_\varepsilon - \check{u}_\varepsilon) \exp(\check{u}_\varepsilon), \quad \check{u}(0) = A \ln \varepsilon, \quad \check{u}_\varepsilon(+\infty) = 0,$$

$$(19) \quad -\Lambda \check{y}_\varepsilon'' = -c_\varepsilon n_0 \hat{m}_\varepsilon', \quad \check{y}_\varepsilon(0) = \hat{y}_\varepsilon(0), \quad \check{y}_\varepsilon(+\infty) = 0,$$

such that $\check{u}_\varepsilon - \hat{u}_\varepsilon = \mathcal{O}(\varepsilon)$ and $\check{y}_\varepsilon - \hat{y}_\varepsilon = \mathcal{O}(\varepsilon)$. We also rescale $M_\varepsilon(x)$,

$$\hat{M}_\varepsilon(\xi) := \int_\xi^{+\infty} \hat{m}_\varepsilon(\xi) d\xi.$$

Let ξ_{vf}^ε the position of the vaporisation front and assume $\xi_{vf}^\varepsilon > 0$, that is some vaporisation occurs after $\xi = 0$. We have $\hat{m}_\varepsilon' = 0$ on $(\xi_{vf}^\varepsilon, +\infty)$, hence \check{y}_ε is linear with limit zero at infinity, therefore $\check{y}_\varepsilon = 0$ on $(\xi_{vf}^\varepsilon, +\infty)$. It follows that

$$-\Lambda \check{y}_\varepsilon(\xi) = +c_\varepsilon n_0 \hat{M}_\varepsilon(\xi).$$

On the other hand, we had for the original rescaled unknown

$$\hat{y}_\varepsilon(0) = \hat{u}_\varepsilon(0) + \Lambda \hat{v}_\varepsilon(0) \ln \varepsilon,$$

because $\hat{u}_\varepsilon(0) = A \ln \varepsilon$, and (Proposition 1 & 2), $0 \leq \hat{v}_\varepsilon \leq \beta_2(\Lambda) |\hat{u}_\varepsilon|$. It follows

$$-\Lambda \check{y}_\varepsilon(0) = -\Lambda \hat{y}_\varepsilon(0) + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon \ln \varepsilon).$$

Since $\hat{M}_\varepsilon(0) = M_\varepsilon(0)/\varepsilon$, we proved

$$M_\varepsilon(0) = \mathcal{O}(\varepsilon \ln \varepsilon).$$

This concludes the proof of Theorem 4. □

Application to the d^2 -law. Assume a vaporisation term of the form

$$\forall \theta_v \leq u \leq 1, \quad \phi(u, m) = g_0 m^\delta, \quad g_0 > 0, \quad 0 \leq \delta < 1.$$

Integrating twice on $[x_\varepsilon, x_{vf}^\varepsilon]$ yields

$$\forall x \geq x_v^\varepsilon, \quad m(x) = \frac{g_0}{cn_0} (1 - \delta)^{1/(1-\delta)} (x_{vf}^\varepsilon - x)^{1/(1-\delta)},$$

$$\forall x \geq x_v, \quad M_\varepsilon(x) = \frac{g_0}{cn_0} \frac{(1 - \delta)^{(2-\delta)/(1-\delta)}}{2 - \delta} (x_{vf}^\varepsilon - x)^{(2-\delta)/(1-\delta)}.$$

The estimate (3.2) above implies, as soon as $x_{vf}^\varepsilon > 0$ that

$$x_{vf}^\varepsilon = \mathcal{O}(\varepsilon \ln \varepsilon)^{(1-\delta)/(2-\delta)}.$$

The so-called “ d^2 -law” states that the rate of variation of the surface area of a vaporising droplet is approximately constant and corresponds to the case $\delta = 1/3$. We let the reader check that it satisfies the hypothesis imposed on the vaporisation term, in particular that complete vaporisation occurs in finite time. The equation above reads

$$x_{vf}^\varepsilon = \mathcal{O}(\varepsilon \ln \varepsilon)^{2/5}.$$

This should be compared with the size of the reaction zone in the case $x_{vf} < 0$, where $x_{vf}^\varepsilon \leq \gamma < 0$ for some γ and ε small enough. In that case, the spray travelling wave has the same velocity as the purely gaseous flame with same temperature in the burnt gas. Also, the analysis of the internal combustion layer is similar to the gaseous case and provides an estimate of $\mathcal{O}(\varepsilon \ln \varepsilon)$ for the size of the reaction zone. This analysis suggests that the internal combustion layer in the presence of droplets might be considerably stretched and provides a quantitative upper estimate in the high activation energy limit. This is in accordance with many observations in the literature. See in particular the numerical experiments in [38] that illustrate the fact that the spray flame consists of a sharp deflagration front followed by a longer afterburn region where the droplets continue to be slowly vaporised.

4. EXTENSIONS

In this section, we state without proofs some easy extensions of our analysis. We consider (i) the case where vaporisation is instantaneous, (ii) the case where the spray is polydisperse, and (iii) other geometries. We refer to [6] for more details and extensions.

Heuristic analysis in the high activation energy limit. A feature specific to our model is the existence of a simple alternative in the high activation energy limit, where it is easy to predict whether the system is in the diffusion or vaporisation controlled regime. Assume the system is in the diffusion controlled regime. This implies that the velocity of the flame is equal to that of the equivalent gaseous flame with same temperature in the burnt gas. The temperature profile in that case is known, hence also the corresponding liquid phase profile, from which one deduces the position x_{vf} where the vaporisation ends. If the vaporisation ends in the preheating zone before the reaction front, i.e. $x_{vf} \leq \bar{x}$, then we are indeed in the diffusion controlled regime. If on the contrary $x_{vf} > \bar{x}$, this is a contradiction with our analysis and the spray flame is in the vaporisation controlled regime. In that case the position \bar{x} of the reaction front obeys the constraint $\bar{x} = x_{vf}$, which corresponds to a unique value of c .

Notice also that thanks to the monotonicity of the vaporisation law we have seen that the transition from diffusion to vaporisation controlled regime is sharp and occurs at a certain $m_u = m_u^*$. As an experience of thought, fix ρ_l , $0 < \rho_l < 1$, and vary m_u from zero to infinity and n_0 accordingly so that $n_0 m_u = \rho_l$ for all values of m_u . Set $v_u = 1 - \rho_l$, and solve for the system in the HAE limit. Our analysis shows that $c = \sqrt{2\mu/\Lambda}$ for $m_u \in (0, m_u^*)$ and $c := c_*(m)$ for $m > m_u^*$, a decreasing function of m . This is in accordance with the numerical experiments shown in [38].

Fast vaporisation. The situation where the vaporisation is very rapid can be modeled by an instantaneous vaporisation (see [7]) leading to a Dirac model for the vaporisation term

$$\phi(u, m) = m_u \delta(x = x_v).$$

Since $\theta_v < \theta_i$, vaporisation ends before the reaction zone at $x_{vf} = x_v < \bar{x}$. Explicit expressions of the profiles can be found.

Polydisperse sprays. All the previous results can be extended in the case of polydisperse sprays, that is the situation where droplets of possibly different sizes are present at a given position. In that case, we need a statistical description of the distribution of size of the droplets as pointed out by WILLIAMS [39, 41]. See also O'ROURKE [30]. For our problem, we assume that the distribution of size of droplets is known in the fresh gases.

Let $dn(t, x, m) = \nu(t, x, m)dm$ denote the density number of droplets at (t, x) with size in the intervall $[m, m + dm]$. The total mass density $\rho_l(t, x)$ of the liquid phase is

$$\rho_l(t, x) = \int_m \nu(t, x, m) m dm.$$

and the function $\nu(t, x, m)$ obeys the conservation equation

$$\partial_t \nu + \partial_m (-\phi(u, m) \nu) = 0.$$

The total vaporisation rate of the liquid phase is

$$\int_m \nu(t, x, m) \phi(u, m) dm,$$

and the corresponding coupled system writes,

$$\begin{aligned} -u'' + cu' &= f(u)v \quad \text{on } \mathbb{R}, \\ -\Lambda v'' + cv' &= -f(u)v + \int_m \nu(t, x, m) \phi(u, m) dm \quad \text{on } \mathbb{R}, \\ c\nu' + \partial_m(-\phi(u, m)\nu) &= 0 \quad \text{on } \mathbb{R}, \\ u(-\infty) &= 0, \quad u(+\infty) = 1, \\ v(-\infty) &= v_u, \quad v(+\infty) = 0, \\ \nu(-\infty) &= \nu_u(m). \end{aligned}$$

where $\nu_u(m)$ is the distribution profile of mass of droplets in the fresh gas.

This polydisperse model admits travelling waves (see also [24]). In the HAE limit, the regime of the flame is determined by the size of the largest droplet in the fresh gas. This result is a consequence of the simplicity of our vaporisation law, where no collective effect, nor influence of the density of the gaseous reactant, nor the fine geometry of the system at small scales were taken into account.

Other geometries. The large activation energy limit analysis can be carried out in other onedimensionnal geometries, such as the anchored flame on the half-line, counter-flow-like configurations, or finally radial geometries in dimension 2 or 3. In each case, one can solve explicitly the problem (provided the expression of the vaporisation law is relatively simple), and it is possible to study the influence of different parameters on the existence of a profile or on the value of the burning rate. Relevant parameters are the typical velocities of injection of the gas and/or droplets, the value of the vaporisation rate, or the space dimension. We refer the reader to [6] for more details.

5. PERSPECTIVES AND CONCLUDING REMARKS

This paper considered the existence of travelling fronts for a simple onedimensionnal thermo-diffusive lean spray flame model. We proved the existence of travelling waves for a class of combustion and vaporisation laws. As far as qualitative results are concerned, the most important part of the paper is the study of the high activation energy limit for the system. The limiting problem involves simple explicit profiles and preserves some important features of the dynamics. Extensions of these results to the cases of fast vaporisation, polydisperse sprays or other onedimensionnal geometries were briefly mentionned.

The present work is a first step towards rigorous derivation of new asymptotic models for spray flames. However, much remains to be done in order to understand the effect of droplets on the dynamics of the flame structure, as observed by many physicists and experimentalists (see [16] and the references therein). Indeed, our

model omits among other things the effects of the latent heat or more complex couplings in the vaporisation law (see [24] for such considerations). It would be very interesting to derive some high activation energy models incorporating these effects, in the hope of deriving explicit expressions of the limiting profiles and combustion rates, as well as extinction limits. Also, a striking consequence of the high activation energy asymptotics is that droplets *cannot* cross the flame front in that limit, but may enter the reaction zone only for large but not infinite values of the activation energy. It would be interesting to derive intermediate asymptotic models providing a precise understanding of the structure and thickening of the combustion region in the presence of vaporising droplets. Ideally, one would hope to recover some of the features of spray flames described in [12, 28, 19, 21, 29].

As far as dynamical phenomena are concerned, the problem of the stability of spray flame systems is crucial. The work [8] presents a mathematical analysis of spray pulsating waves. See also [27, 6, 5]. As a second step we have in mind the problem of acoustic instabilities in spray flame systems. In this direction, let us mention the work of Clavin and Sun [7]. Their mathematical analysis relies on the possibility of deriving explicit expressions of the solutions to the problem. This is only possible in certain asymptotic limits. For that, the authors consider the large activation energy limit for combustion phenomena and assume an instantaneous vaporisation of the droplets. As a consequence, both the combustion and the vaporisation zones reduce to infinitely small regions, whose internal layer is analysed.

However, assuming that the vaporisation zone is very small compared to the preheating zone is a very restrictive assumption in many applications where the droplets can spread into the preheating zone, approach the combustion zone, or enter the combustion zone. We have shown in the present paper that it is possible to analyse such situations where the droplets approach or reach the reaction front, and where the slowly vaporising droplets induce a dramatic change of the combustion rate. We hope that our work will motivate new studies carrying out rigorous mathematical analysis of asymptotic models for spray flames.

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